ON THE TURBULENT BOUNDARY LAYER OF AN IMPERFECT GAS

(O TURBULENTNOM POGRANICHNOM SLOE NESOVERSHENNOGO SZHIMAEMOGO GAZA)

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We consider the derivation of the equation of the turbulent boundary layer of an imperfect gas (e.g., dissociated air). We show that the relations which hold for an ideal gas and which are consequences of the equations of motion, continuity and energy only can be generalized to include the case of an imperfect gas by formally replacing temperature with enthalpy. We give examples of such relations.

1. Boundary layer equations. We consider the system of equations of an averaged plane turbulent flow of a real gas. We first look more closely at the energy equation. This equation is obtained by taking the time average of the energy equation of the actual motion which it is convenient to write in this case in the following form* (the gas is supposed transparent):

$$\frac{\partial}{\partial t}(\rho I) + \frac{\partial}{\partial x}(\rho u I) + \frac{\partial}{\partial y}(\rho v I) = \frac{1}{J}\frac{\partial p}{\partial t} \qquad \left(I = i + \frac{u^2+v^2}{2J}, \quad i = e + \frac{1}{J}\frac{p}{\rho}\right) \tag{1.1}$$

Here x and y are the Cartesian coordinates of a point, t is time, ρ is density, p is pressure, I is the stagnation enthalpy, i is enthalpy, e is internal energy and J is the mechanical equivalent of heat.

(*Note.* In order to simplify computations we omitted in Equation (1.1) terms due to molecular heat conduction and viscosity. This is justified

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* Strictly speaking one should consider the equation of energy for a three-dimensional flow for the simple reason that the actual motion of the gas is three-dimensional. However, in case of a plane averaged flow this does not affect the final result.

by the fact that, after averaging, these terms turn out to be small compared with the corresponding turbulent characteristics. Equation (1.1) holds for an imperfect gas, i.e. when the specific heats c_p and c_v depend on temperature and pressure and the equation of state is arbitrary. In particular, (1.1) is valid for equilibrium dissociation of air or other mixtures of gases.)

To obtain the energy equation of an averaged turbulent motion one must write the functions in (1.1) as sums of an averaged and a fluctuating component

$$u = \bar{u} + u', \qquad v = \bar{v} + v', \qquad i = \bar{i} + i'$$

etc., and average with respect to time. After carrying out all the necessary transformations using the averaged continuity equation [cf. Equation (1.11)] and disregarding fluctuating terms of order higher than two we get

$$\rho\left(\frac{\partial I^\circ}{\partial t} + u^\circ \frac{\partial I^\circ}{\partial x} + v^\circ \frac{\partial I^\circ}{\partial y}\right) = \frac{1}{2J}\frac{\partial}{\partial t}(\bar{p}_{xx} + 2\bar{p} + \bar{p}_{yy}) + \frac{1}{2J}\left[\frac{\partial}{\partial x}(3u^\circ \bar{p}_{xx} + \right.$$
$$\left. + 2v^\circ \bar{\tau}_{xy} + u^\circ \bar{p}_{yy}) + \frac{\partial}{\partial y}(3v^\circ \bar{p}_{yy} + 2u^\circ \bar{\tau}_{xy} + v^\circ \bar{p}_{xx})\right] - \frac{\partial \bar{q}_x}{\partial x} - \frac{\partial \bar{q}_y}{\partial y} \quad (1.2)$$

$$u^\circ = \frac{\overline{\rho u}}{\bar{\rho}} = \bar{u}\left[1 + \frac{\overline{\rho' u'}}{\bar{\rho}\,\bar{u}}\right], \qquad v^\circ = \frac{\overline{\rho v}}{\bar{\rho}} = \bar{v}\left[1 + \frac{\overline{\rho' v'}}{\bar{\rho}\,\bar{v}}\right],$$

$$i^\circ = \frac{\overline{\rho i}}{\bar{\rho}} = \bar{i}\left[1 + \frac{\overline{\rho' i'}}{\bar{\rho}\,\bar{i}}\right], \qquad I^\circ = i^\circ + \frac{u^{\circ 2} + v^{\circ 2}}{2J}$$

$$\bar{p}_{xx} = -\bar{\rho}\,\overline{u'^2}, \qquad \bar{\tau}_{xy} = -\bar{\rho}\,\overline{u'v'}, \qquad \bar{p}_{yy} = -\bar{\rho}\,\overline{v'^2} \qquad \text{(turbulent stress)}$$

$$\bar{q}_x = \bar{\rho}\,\overline{u'i'}, \qquad \bar{q}_y = \bar{\rho}\,\overline{v'i'} \qquad \text{(turbulent heat flux)}$$

Here u°, v°, i° represent the averaged components of velocity and enthalpy in the flow of a compressible gas and I° the average stagnation enthalpy.

We note that with the given definition of the mean quantities u°, v° i°, density fluctuations do not enter explicitly in Equation (1.2). The velocities u°, v° are ratios of the appropriate averaged divergence to the mean density, and the enthalpy i° is the ratio of the specific thermal capacity to the mean density.

An important instance of the flow of an imperfect gas is the flow in a boundary layer. To obtain the energy equation in the boundary layer we must neglect in (1.2) quantities of order v° in comparison with quantities of order u°, and derivatives with respect to x in comparison with derivatives with respect to y (the x-axis is supposed directed along the body profile). Also in the boundary layer equation we must take into account terms due to molecular viscosity and heat conduction, since these terms are significant in the immediate vicinity of the rigid surface where the turbulent fluctuations die out. We have

$$\bar{\rho}\left(\frac{\partial I^\circ}{\partial t}+u^\circ\frac{\partial I^\circ}{\partial x}+v^\circ\frac{\partial I^\circ}{\partial y}\right)= \qquad \left(I^\circ=i^\circ+\frac{u^{\circ 2}}{2J}\right)$$

$$=\frac{1}{2J}\frac{\partial}{\partial t}(\bar{p}_{xx}+2\bar{p}+\bar{p}_{yy})+\frac{1}{J}\frac{\partial}{\partial y}\left(\overline{\mu u\frac{\partial u}{\partial y}}+u^\circ\bar{\tau}_{xy}\right)+\frac{\partial}{\partial y}\left(\overline{\eta\frac{\partial T}{\partial y}}-\bar{q}_y\right) \quad (1.3)$$

Here T is the temperature, μ is the viscosity coefficient, and η is the coefficient of heat conduction. We represent the turbulent stress $\bar{\tau}_{xy}$ and the turbulent heat flux $\bar{q}_y$ (by analogy with the corresponding molecular characteristics) in the following form

$$\bar{\tau}_{xy}=\varepsilon\frac{\partial u^\circ}{\partial y}, \qquad \bar{q}_y=-\eta_t\frac{\partial T^\circ}{\partial y} \qquad \left(T^\circ=\frac{\overline{\rho T}}{\bar{\rho}}=\bar{T}\left[1+\frac{\overline{\rho' T'}}{\bar{\rho}\bar{T}}\right]\right)$$

where ϵ and η_t are the coefficients of turbulent viscosity and turbulent heat conduction. The relations just written out can be regarded as definitions of ϵ and η_t. In the expression for $\bar{q}_y$ we use the mean temperature T° because in the case of an ideal gas (c_p = const) this quantity appears on the left-hand side of Equation (1.3) in place of the enthalpy i° (i.e. for c_p = const, $i^\circ = c_p T^\circ$) and it can therefore be viewed as an extension of the notion of average temperature to the case of a compressible gas. A further simplification consists in the replacement of the quantities

$$\overline{\mu u\,\partial u/\partial y}, \qquad \overline{\eta\,\partial T/\partial y} \quad \text{by} \quad \mu u^\circ\,\partial u^\circ/\partial y, \qquad \eta\,\partial T^\circ/\partial y$$

This is justified by the fact that in the laminar sublayer close to the wall where these terms are significant there are no turbulent fluctuations. Therefore

$$\overline{\mu u\,\partial u/\partial y}=\mu u^\circ\,\partial u^\circ/\partial y=\mu u\,\partial u/\partial y, \qquad \overline{\eta\,\partial T/\partial y}=\eta\,\partial T^\circ/\partial y=\eta\,\partial T/\partial y$$

However, it should be noted that in the boundary layer some distance away from the wall there always exists a "transition" domain in which the coefficients of molecular and turbulent viscosity and of heat conduction are of the same order. In this domain the above simplification is not valid. Therefore the equations given below, namely, the energy equation and the equation of motion corresponding to the generally accepted form in which this equation is stated (utilizing this simplification) are based on an assumed division of the boundary layer into two domains with different flow regimes, a turbulent outer layer and a laminar sublayer. We have

$$\bar{\bar{\rho}}\left(\frac{\partial I^\circ}{\partial t} + u^\circ \frac{\partial I^\circ}{\partial x} + v^\circ \frac{\partial I^\circ}{\partial y}\right) =$$
$$= \frac{1}{2J}\frac{\partial}{\partial t}(\bar{p}_{xx} + 2\bar{p} + \bar{p}_{vv}) + \frac{\partial}{\partial y}\left[(\eta + \eta_t)\frac{\partial T^\circ}{\partial y}\right] + \frac{1}{J}\frac{\partial}{\partial y}\left[(\mu + \varepsilon)\, u^\circ \frac{\partial u^\circ}{\partial y}\right]$$

Assuming the average enthalpy i° to be a function of the mean temperature and pressure, $i^\circ = i^\circ(T^\circ, \bar{p})$, we have

$$\frac{\partial i^\circ}{\partial y} = \frac{\partial i^\circ}{\partial T^\circ}\frac{\partial T^\circ}{\partial y} + \frac{\partial i^\circ}{\partial \bar{p}}\frac{\partial \bar{p}}{\partial y}$$

If we define the mean specific heat c_p° by means of the relation $\partial i^\circ(T^\circ, p)/\partial T^\circ$ and observe that in the boundary layer $\partial \bar{p}/\partial y = 0$, we obtain the following formula which is valid in the general case only for an ideal gas:

$$\frac{\partial i^\circ}{\partial y} = c_p{}^\circ \frac{\partial T^\circ}{\partial y} \tag{1.4}$$

Using (1.4) and introducing the Prandtl number $P = \mu c_p/\eta$ and the Prandtl number for turbulent mixing $P_t = \epsilon c_p{}^\circ/\eta_t$ we obtain the energy equation for the turbulent boundary layer

$$\bar{\rho}\left(\frac{\partial I^\circ}{\partial t} + u^\circ \frac{\partial I^\circ}{\partial x} + v^\circ \frac{\partial I^\circ}{\partial y}\right) = \frac{1}{2J}\frac{\partial}{\partial t}(\bar{p}_{xx} + 2\bar{p} + \bar{p}_y) +$$
$$+ \frac{\partial}{\partial y}\left[(\mu + \varepsilon)\frac{\partial I^\circ}{\partial y}\right] + \frac{\partial}{\partial y}\left[\left(\frac{1}{P} - 1\right)\mu \frac{\partial i^\circ}{\partial y}\right] + \frac{\partial}{\partial y}\left[\left(\frac{1}{P_t} - 1\right)\varepsilon \frac{\partial i^\circ}{\partial y}\right] \tag{1.5}$$

We note that (1.5) differs from the energy equation for an ideal gas only in that in place of the mean temperature it contains the mean enthalpy. The energy equation for a laminar boundary layer is a special case of (1.5). Next we consider the equation of state. In the case of pressures well below critical it can be written (for actual quantities) in the form

$$pm = K\rho T \tag{1.6}$$

where m is the molecular weight of the gas, and K is the universal gas constant.

If we average both parts of (1.6) with respect to time we get

$$\bar{p}m^\circ = K\bar{\rho}T^\circ \qquad \left(m^\circ = \frac{\overline{pm}}{\bar{p}} = \bar{m}\left[1 + \frac{\overline{p'm'}}{\bar{p}\,\bar{m}}\right]\right) \tag{1.7}$$

Thus, with an appropriate definition of mean molecular weight we find that the equation of state connecting the mean pressure, density and temperature - all of which appear in (1.5) - retains its usual form.

The molecular weight of a gaseous mixture (e.g. dissociated air) is connected with the molecular weight of the individual components m_k by means of the well-known formula

$$m = \left(\sum \frac{M_k}{m_k}\right)^{-1} \tag{1.8}$$

where M_k is the mass fraction of the kth component of the mixture. Taking the time average of both sides of (1.8) we get

$$\overline{m} = \overline{\left(\sum \frac{M_k}{m_k}\right)^{-1}} \tag{1.9}$$

If we compare the expression (1.7) for m° with (1.9) we see that the dependence of the mean molecular weight m° on the quantities m_k in case of a turbulent flow differs from the corresponding dependence under static conditions. Its explicit form can be established only by means of relations from the theory of turbulence which connect the mean quantities with mean values of the fluctuations, i.e. it depends on the characteristics of the flow. The same can be said about the connection between mean enthalpy i° and temperature T°. When prescribing the values of these quantities in the turbulent boundary layer of dissociated air it is not possible, in general, to make use of tables based on the assumption of equilibrium of dissociation which fail to take into account the special features of the motion of gas.

The equations of motion and continuity do not change form when we go from an ideal gas to an imperfect gas. These equations are:

$$\bar{\rho}\left(\frac{\partial u^\circ}{\partial t} + u^\circ \frac{\partial u^\circ}{\partial x} + v^\circ \frac{\partial u^\circ}{\partial y}\right) = -\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial y}\left[(\mu + \varepsilon)\frac{\partial u^\circ}{\partial y}\right] \tag{1.10}$$

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x}(\bar{\rho} u^\circ) + \frac{\partial}{\partial y}(\bar{\rho} v^\circ) = 0 \tag{1.11}$$

From what we have said it follows that the system of equations of the turbulent boundary layer of an imperfect gas is more complicated than the corresponding system for c_p = const and m = const and contains a greater number of variables.

However, relations which hold for an ideal gas and are consequences of the equations of energy and continuity alone can be easily extended to the case of an imperfect gas by formally replacing temperature with enthalpy. Examples of such relations are given below.

2. Integrals of the energy equation. Connection between heat conduction and friction. In the following we drop the symbol for taking averages. This is done to simplify notation. The index w denotes values of quantities on the wall and ∞ values of quantities outside the boundary layer.

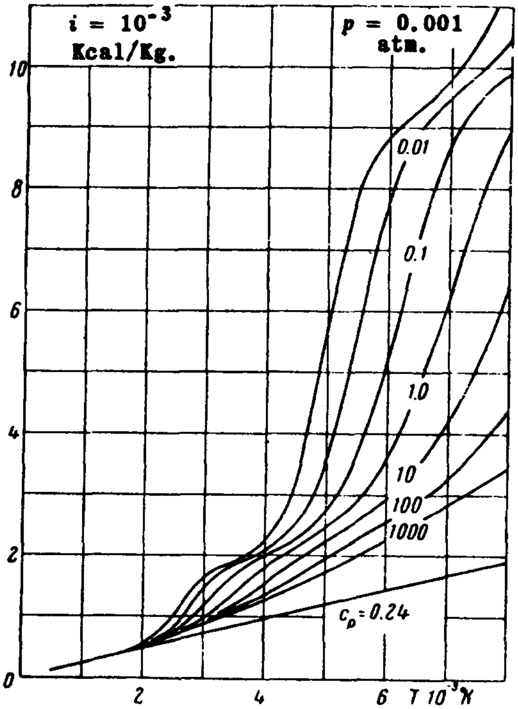
For stationary mean motion $P = P_t = 1$, in the absence of heat exchange between the body and the gas

$$\left(\frac{\partial T}{\partial y}\right)_w = \left(\frac{\partial i}{\partial y}\right)_w = \left(\frac{\partial I}{\partial y}\right)_w = 0$$

and for arbitrary distribution of pressure over the profile of the body we obtain, on the basis of (1.5),

$$I = \text{const} = I_\infty \tag{2.1}$$

In spite of the fact that i_w = const, the temperature of the surface of the body does not, in general, remain fixed but varies in accordance



with the form of the function $i(T, p)$. In the case of dissociated air which satisfies approximately the conditions of the problem under consideration, the temperature of the wall T_w turns out to be low compared with the values obtained under the assumption that air is at all temperatures an ideal gas whose specific heat c_p is 0.24 Kcal/Kg. degr. This is illustrated in the figure which shows the dependence of enthalpy of air i on its temperature $T°$ K and pressure (p in atm.) under the assumption of equilibrium of dissociation [1]. The figure also shows the relation $i = c_p T$ for an ideal gas with $c_p = 0.24$ Kcal/Kg. degr.

We now consider the formula for the hydrodynamical model of heat conduction (the Reynolds model). In the case of stationary mean motion for $p =$ const, $P = P_t = 1$, $T_w =$ const, Equations (1.5), (1.10) and the boundary conditions

$$u = 0 \text{ for } y = 0, \qquad u \to U \text{ for } y \to \infty$$

we obtain a linear relation connecting velocity and the stagnation enthalpy:

$$\frac{I - I_w}{I_\infty - I_w} = \frac{u}{U} \tag{2.2}$$

(2.2) implies the well-known connection between the Nusselt number N_w, the Reynolds number R_w and the coefficient of friction c_f:

$$N_w = \tfrac{1}{2} R_w c_f$$

Here (L is a characteristic length)

$$N_w = \frac{Q_w L}{\eta_w (I_w - I_\infty)}, \qquad Q_w = -\eta_w \left(\frac{\partial i}{\partial y}\right)_w, \qquad R_w = \frac{\rho_\infty U L}{\mu_w}, \qquad c_f = \frac{2\tau_w}{\rho_\infty U^2}$$

If we insert in the expression for the Nusselt number the heat flux $\bar{q}_w = -\eta_w (\partial T/\partial y)_w$ by means of the relation (1.4) and the values of the coefficient of heat conduction η_∞ and the specific heat $c_{p\infty}$ in the unperturbed stream we obtain

$$N_w = N \frac{P_w}{P_\infty} \frac{\mu_\infty}{\mu_w} \qquad \left(N = \frac{q_w c_{p\infty} L}{\eta_\infty (I_w - I_\infty)}\right) \tag{2.3}$$

We put the Reynolds number in the following form

$$R_w = R \frac{\mu_\infty}{\mu_w} \qquad \left(R = \frac{\rho_\infty U L}{\mu_\infty}\right)$$

In the case at hand we assume $P = P_t = 1$ so that

$$N = \tfrac{1}{2} R c_f \tag{2.4}$$

We see that if the Nusselt number is defined by (2.3), then the formula for the Reynolds model retains its usual form in case of an imperfect gas. Certain approximate relations based only on the equations of motion, continuity and energy admit of similar extension.

The approximate connection between the stagnation temperature and the velocity in a stationary turbulent boundary layer near a plate for $P \neq P_t \neq 1$, p = const, T_w = const can be represented in the turbulent outer layer of the boundary layer by

$$I - I_\infty = \frac{q_w P_t U}{\tau_w}(1 - \bar{u}) - \frac{U^2}{2J}(1 - P_t)\,[(1 - \bar{u}^2) - A\,(1 - \bar{u}^4)]$$

and in the laminar sublayer by

$$I = I_w - \frac{q_w P U}{\tau_w}\,\bar{u} + \frac{U^2}{2J}(1 - P)\,(\bar{u}^2 - A\bar{u}^4)$$

Here

$$\bar{u} = \frac{u}{U}, \qquad A = \int_0^\infty \frac{\rho}{\rho_\infty}\,\bar{u}\,(1 - \bar{u}^2)\,dy \Big/ \int_0^\infty \frac{\rho}{\rho_\infty}\,\bar{u}\,(1 - \bar{u}^4)\,dy$$

In case of a turbulent boundary layer with $\bar{u} \approx y^{1/n}$, $n = 7 \div 8$, computations yield $A = 0.59 \div 0.60$. In the given formulas the terms proportional to $\bar{u}$, $\bar{u}^2$, $\bar{u}^3$ represent the first terms in the expansion of $I(\bar{u})$ in the vicinity of the wall. The coefficient of $\bar{u}^4$ is chosen so that the complete profile of the stagnation enthalpy for $q_w = 0$ satisfies the energy integral

$$\int_0^\infty \frac{\rho}{\rho_\infty}\,\bar{u}\left(1 - \frac{I}{I_\infty}\right) dy = 0$$

When these relations are applied at the boundary of the laminar sublayer we obtain

$$N = \tfrac{1}{2} R c_f \left[\frac{P_t}{P} + \bar{u}_\wedge\left(1 - \frac{P_t}{P}\right)\right]^{-1} \qquad \left(N = \frac{q_w c_{p\infty} L}{\eta_w\,[I_w - I_{w0}]}\right) \quad (2.5)$$

Here I_{w0} is the enthalpy of the gas at the wall in the absence of heat exchange between the wall and the gas; $\bar{u}_\wedge = u_\wedge / U$ is the ratio of the velocity on the boundary of the laminar sublayer to the velocity of the unperturbed stream. According to the semi-empirical theory of turbulence this quantity is connected with the coefficient of friction

$$\bar{u}_\wedge = \frac{\alpha}{\sqrt{P}}\sqrt{\frac{c_f}{2}\,\frac{\rho_\infty}{\rho_w}}$$

Here α is an empirical constant of the laminar sublayer (experiments with incompressible fluids give $\alpha = 11.5$).

The quantity I_{w0} for $P \neq 1$ and $P_t \neq 1$ does not coincide with the stagnation enthalpy of the unperturbed stream. The difference is determined by the value of the "coefficient of restoration of enthalpy"

$$\vartheta \equiv \frac{I_{w0} - i_\infty}{I_\infty - i_\infty} = 1 - \{(1 - P_t)\,[(1 - \bar{u}_\wedge^2) - A\,(1 - \bar{u}_\wedge^4)] + (1 - P)(\bar{u}_\wedge^2 - A\bar{u}_\wedge^4)\} \quad (2.6)$$

We note that for $P = P_t \neq 1$ the velocity on the boundary of the laminar sublayer does not appear in (2.5) and (2.6). In this case

$$N = \tfrac{1}{2} Rc_f, \qquad \vartheta = 1 - (1 - P)(1 - A) \tag{2.7}$$

Formula (2.7) differs from (2.4) only in that the quantity I_{w0} which appears in the Nusselt number is not equal to I_∞.

The relations (2.5) and (2.6) were obtained under the assumption that P and P_t are constant. In the case of a dissociated gas (e.g. air) the Prandtl number P depends on the extent of dissociation of the gas, i.e. on its temperature. However, within the limits of the laminar sublayer where the value of the parameter P is significant, the variation of temperature is small. As for the Prandtl number of turbulent mixing P_t it, as well as other characteristics of turbulence, does not, apparently, depend on the chemical nature of the gas or liquid and can therefore be taken equal to that number for an ideal gas. It should be noted that at the moment our knowledge concerning the Prandtl number in the case of turbulent mixing is far from complete. Prandtl [2] gives the value for the parameter $P_t \approx 0.7$, i.e. for air $P \approx P_t$.

When we take into account the difference of the values of the parameter P at the wall (in the laminar sublayer) and in the unperturbed flow, Formula (2.5) can be written as follows:

$$N = \tfrac{1}{2} Rc_f \left[\frac{P_t}{P_\infty} + \bar{u}_\wedge \left(\frac{P_w}{P_\infty} - \frac{P_t}{P_\infty}\right)\right]^{-1} \tag{2.8}$$

From the specified data it is clear that when studying problems of heat exchange between the wall and the stream of an imperfect gas, it is more convenient to use the generalized Nusselt number [cf. Equation (2.5)] rather than the usual

$$N = \frac{q_w L}{\eta_\infty (T_w - T_{w0})}$$

for then certain relations which follow from the equations of motion, continuity and energy do not change form. It is expedient to replace the temperature recovery coefficient by the enthalpy recovery coefficient. These points should be kept in mind when processing experimental data.

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